

**Warsaw University
of Technology**



**Faculty of Power and
Aeronautical Engineering**

WARSAW UNIVERSITY OF TECHNOLOGY

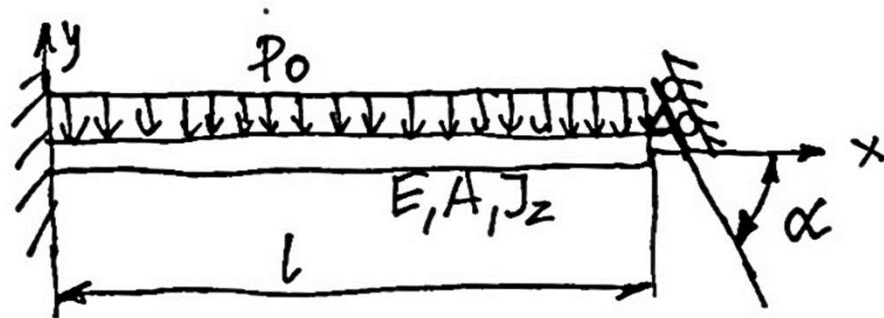
Institute of Aeronautics and Applied Mechanics

Finite element method (FEM)

Plane frame. Example 2.

05.2021

EXAMPLE. BUILD A FE MODEL USING 2D FRAME ELEMENT.
FIND UNKNOWN DISPLACEMENTS AND REACTIONS.

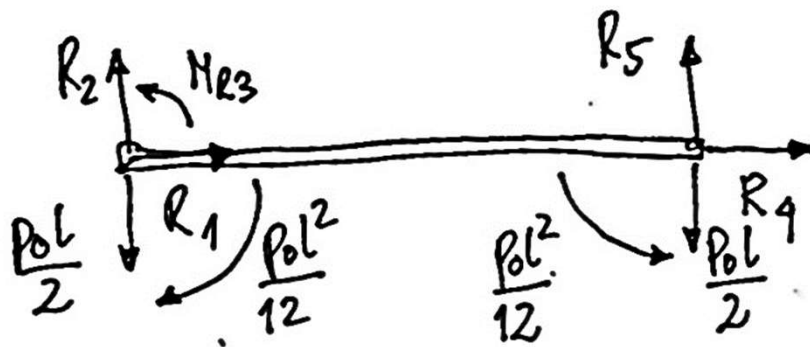


NDOF=6

10)



$$\{q\}_{6 \times 1} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$



$$\{F\}_{6 \times 1} = \begin{Bmatrix} R_1 \\ R_2 - \frac{Pol}{2} \\ M_{23} - \frac{Pol^2}{12} \\ R_4 \\ R_5 - \frac{Pol}{2} \\ \frac{Pol^2}{12} \end{Bmatrix}$$

$$[K]_{6 \times 6} = \begin{bmatrix} a & 0 & 0 & -a & 0 & 0 \\ 0 & b & d & 0 & -b & d \\ 0 & d & m & 0 & -d & r \\ -a & 0 & 0 & a & 0 & 0 \\ 0 & -b & -d & 0 & b & -d \\ 0 & d & r & 0 & -d & m \end{bmatrix}$$

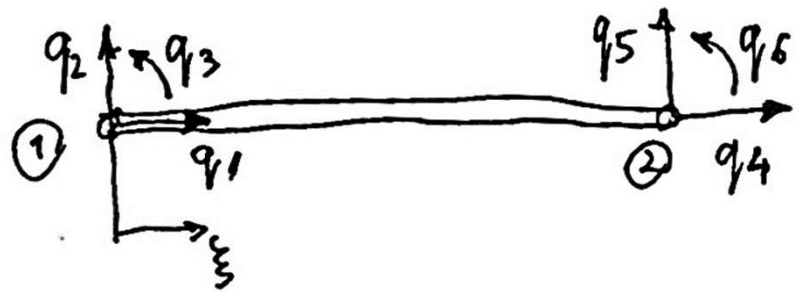
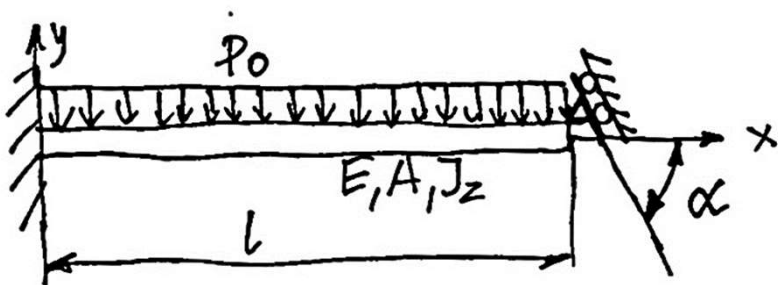
$$a = \frac{EA}{L}$$

$$b = \frac{12EJ_2}{L^3}$$

$$d = \frac{6EJ_2}{L^2}$$

$$m = \frac{4EJ_2}{L}$$

$$r = \frac{2EJ_2}{L}$$



boundary conditions

$$q_1 = 0$$

$$q_2 = 0$$

$$q_3 = 0$$

②



(-) q_4 ②

$$\tan \alpha = \frac{q_5}{-q_4}$$

\Rightarrow

$$q_5 = -q_4 \cdot \tan \alpha$$

NOF = 4

$$N = \text{NDOF} - \text{NOF} = 5 - 4 = 2$$

unknown (independent) degrees of freedom :

$$\begin{matrix} \{q\} \\ \text{NDOF} \times 1 \end{matrix} = \begin{matrix} [C] \\ \text{NDOF} \times N \end{matrix} \cdot \begin{matrix} \{q\} \\ N \times 1 \end{matrix} ; \quad \begin{matrix} \{q\} \\ 6 \times 1 \end{matrix} = \begin{matrix} [C] \\ 6 \times 2 \end{matrix} \cdot \begin{matrix} \{q\} \\ 2 \times 1 \end{matrix}$$

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -\text{tg}\alpha & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} q_4 \\ q_6 \end{Bmatrix} ;$$

$$\begin{matrix} \{q\} \\ 1 \times 6 \end{matrix} = \begin{matrix} \{q\} \\ 1 \times 2 \end{matrix} \cdot \begin{matrix} [C]^T \\ 2 \times 6 \end{matrix}$$

$$\begin{matrix} [C]^T \\ 2 \times 6 \end{matrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & -\text{tg}\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{tg}\alpha = t$$

TOTAL POTENTIAL ENERGY:

$$V = \frac{1}{2} \underset{1 \times 6}{Lq} \underset{6 \times 6}{[K]} \underset{6 \times 1}{\{q\}} - \underset{1 \times 6}{Lq} \underset{6 \times 1}{\{F\}} =$$

$$= \frac{1}{2} \underset{1 \times 2}{Lq} \underset{2 \times 6}{[C]}^T \underset{6 \times 6}{[K]} \underset{6 \times 2}{[C]} \underset{2 \times 1}{\{q\}} - \underset{1 \times 2}{Lq} \underset{2 \times 6}{[C]}^T \underset{6 \times 1}{\{F\}} =$$

$$= \frac{1}{2} \underset{1 \times 2}{Lq} \underset{2 \times 2}{[K]} \underset{2 \times 1}{\{q\}} - \underset{1 \times 2}{Lq} \underset{2 \times 1}{\{F\}} \Rightarrow \underset{2 \times 2}{[K]} \underset{2 \times 1}{\{q\}} = \underset{2 \times 1}{\{F\}}$$

(boundary conditions included)

$$[K] \cdot [C] = \begin{bmatrix} -a & 0 \\ bt & d \\ td & r \\ a & 0 \\ -bt & -d \\ td & m \end{bmatrix}$$

$0 \times 6 \quad 6 \times 2$

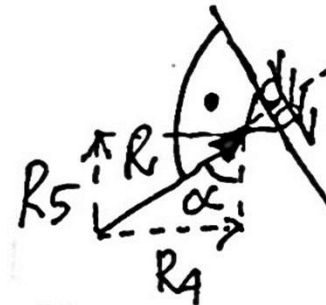
$$[C]^T \cdot ([K] \cdot [C]) = \begin{bmatrix} a+bt^2 & | & td \\ td & | & m \end{bmatrix}$$

$2 \times 6 \quad 6 \times 6 \quad 6 \times 2$
 $[K]$
 2×2

$$[C]^T \cdot \{F\} = \begin{Bmatrix} R_4 - t(R_5 - \frac{P_0 L}{2}) \\ \frac{P_0 L^2}{12} \end{Bmatrix} = \begin{Bmatrix} \frac{P_0 L}{2} \cdot t \\ \frac{P_0 L^2}{12} \end{Bmatrix}$$

$$\frac{R_4}{R_5} = \tan \alpha = t$$

$$R_4 - t \cdot R_5 = 0$$



$$\begin{cases} (a + bt^2) q_4 + td \cdot q_6 = \frac{P_0 L}{2} \cdot t \\ td \cdot q_4 + m \cdot q_6 = \frac{P_0 L^2}{12} \end{cases}$$

$$\text{II} \rightarrow q_6 = \frac{\frac{P_0 L^2}{12} - td \cdot q_4}{m}$$

$$\text{I} \rightarrow (a + bt^2) q_4 + \frac{t \cdot d}{m} \left(\frac{P_0 L^2}{12} - td \cdot q_4 \right) = \frac{P_0 L}{2} \cdot t$$

$$\left(a + bt^2 - \frac{t^2 d^2}{m} \right) q_4 = \frac{P_0 L}{2} t - \frac{P_0 L^2 d}{12 m} \cdot t$$

$$q_4 = \frac{\frac{P_0 L t}{12} \left(6 - \frac{L d}{m} \right)}{a + bt^2 - \frac{d^2}{m} t^2} = \frac{\frac{P_0 L t}{12} \left(6 - \frac{L \cdot 6 E J_2 \cdot L}{L^2 \cdot 4 E J_2} \right)}{\frac{EA}{L} + \frac{12 E J_2}{L^3} t^2 - \frac{36 E^2 J_2^2 \cdot L}{L^4 \cdot 4 E J_2} t^2}$$

$$= \frac{\frac{3}{8} P_0 L t}{\frac{EA}{L} + \frac{3 E J_2}{L^3} t^2} = \frac{\frac{3}{8 t} P_0 L}{\frac{EA}{L t^2} + \frac{3 E J_2}{L^3}}$$

$$q_5 = -q_4 \cdot t = -\frac{\frac{3}{8} p_0 L t^2}{\frac{EA}{L} + \frac{3EJ_z}{L^3} t^2} = \frac{-\frac{3}{8} p_0 L}{\frac{EA}{L t^2} + \frac{3EJ_z}{L^3}}$$

$$\alpha = 0^\circ : q_4 = 0, q_5 = 0$$

$$\alpha = 90^\circ \rightarrow t \rightarrow \infty : q_4 = 0, q_5 = -\frac{3}{24} \frac{p_0 L^4}{EJ_z}$$

$$q_6 = \frac{\frac{P_0 L^2}{12} - t \cdot d \cdot \frac{3}{8t} \cdot P_0 \cdot L}{m} / \left(\frac{EA}{Lt^2} + \frac{3EJ_2}{L^3} \right) =$$

$$= \frac{\left(\frac{P_0 L^2}{12} - \frac{3}{8} P_0 L \cdot \frac{6EJ_2}{L^2 \left(\frac{EA}{Lt^2} + \frac{3EJ_2}{L^3} \right)} \right) L}{4EJ_2} =$$

$$= \frac{\frac{P_0 L^3}{12} - \frac{18}{8} P_0 \cdot \frac{1}{\frac{A}{J_2 Lt^2} + \frac{3}{L^3}}}{4EJ_2} = \frac{\frac{P_0 L^3}{12} - \frac{9}{4} \cdot \frac{P_0 L^3}{\frac{AL^2}{J_2 t^2} + 3}}{4EJ_2} =$$

$$= \frac{P_0 L^3}{48EJ_2} \left(1 - \frac{27}{\frac{AL^2}{J_2 t^2} + 3} \right)$$

REACTIONS:

$$\underset{6 \times 6}{[K]} \cdot \underset{6 \times 1}{\{q\}} = \underset{6 \times 1}{\{F\}}$$

$$-a \cdot q_4 = R_1$$

$$-b \cdot q_5 + d \cdot q_6 = R_2 - \frac{P_0 L}{2}$$

$$-d \cdot q_5 + r \cdot q_6 = M_{e3} - \frac{P_0 L^2}{12}$$

$$a \cdot q_4 = R_4$$

$$b \cdot q_5 - d \cdot q_6 = R_5 - \frac{P_0 L}{2}$$

$$R_1 = -\frac{EA}{L} \cdot q_4$$

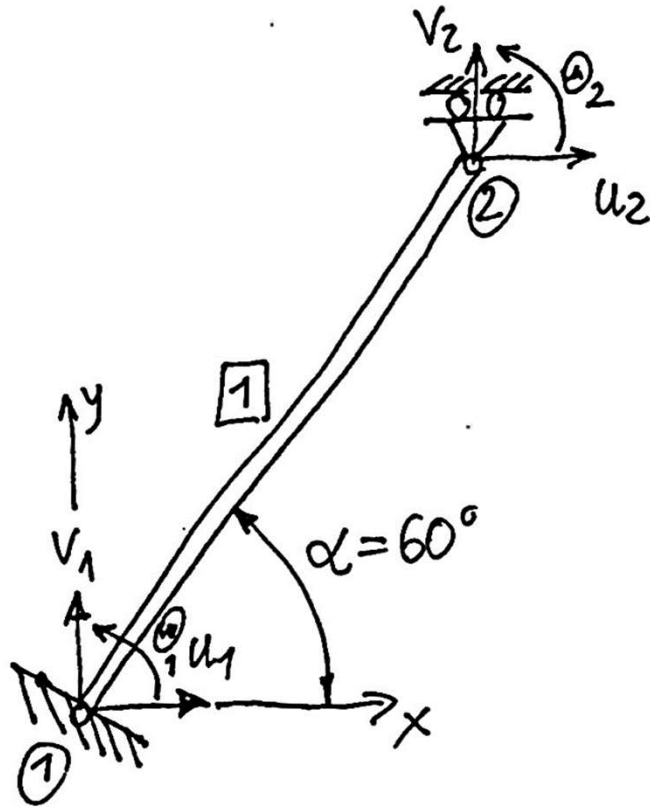
$$R_2 = -\frac{12EJ_2}{L^3} \cdot q_5 + \frac{6EJ_2}{L^2} \cdot q_6 + \frac{P_0 L}{2}$$

$$M_{e3} = -\frac{6EJ_2}{L^2} \cdot q_5 + \frac{2EJ_2}{L} \cdot q_6 + \frac{P_0 L^2}{12}$$

$$R_4 = \frac{EA}{L} \cdot q_4$$

$$R_5 = \frac{12EJ_2}{L^3} q_5 - \frac{6EJ_2}{L^2} \cdot q_6 + \frac{P_0 L}{2}$$

2^a)

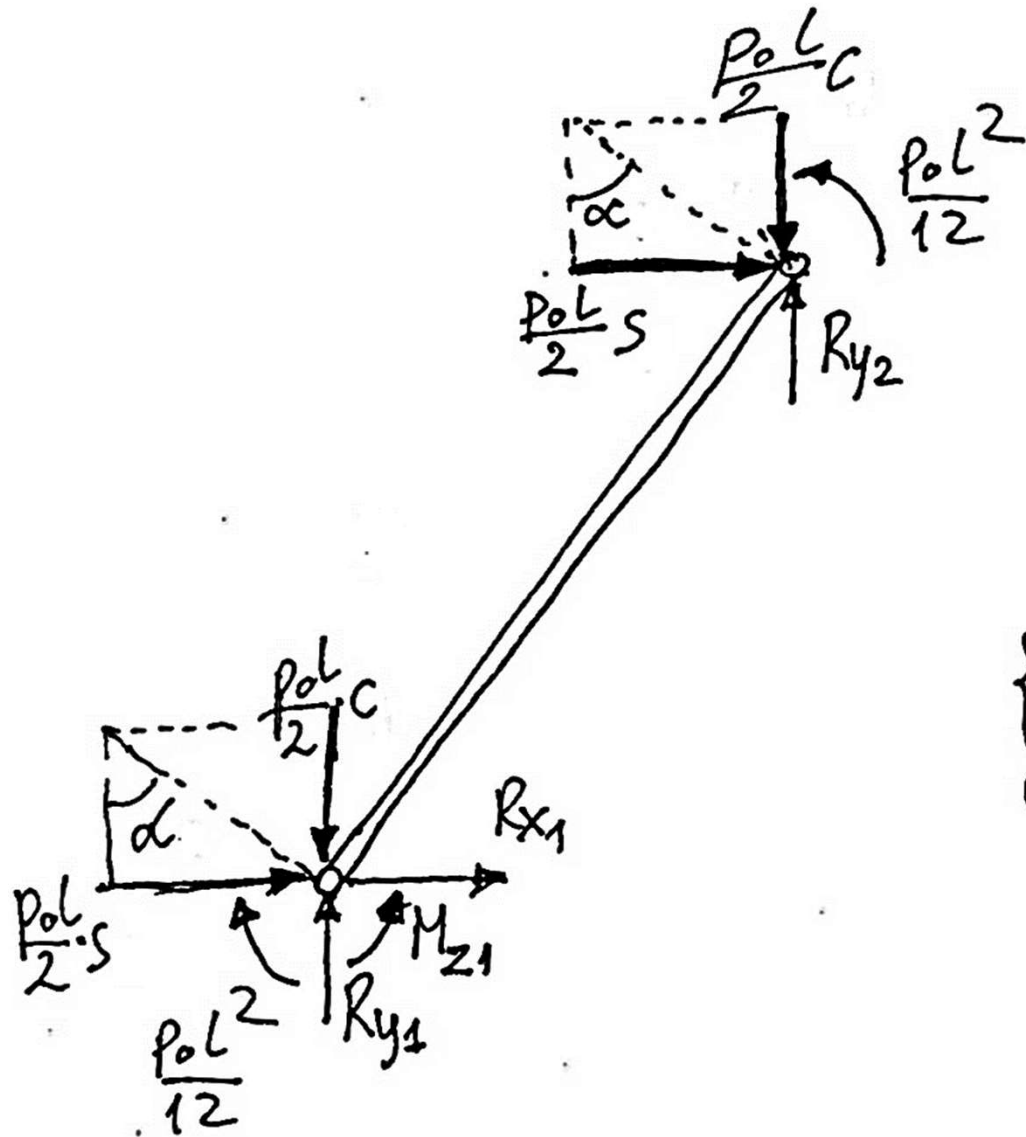


$$C = \frac{1}{2} \quad t = \frac{s}{c} = \sqrt{3}$$

$$s = \frac{\sqrt{3}}{2}$$

$$L \underset{1 \times 6}{\mathcal{Q}} \underset{1}{\mathcal{J}} = L \langle u_1, v_1, \theta_1, u_2, v_2, \theta_2 \rangle$$

$$L \underset{1 \times 6}{\mathcal{Q}} \underset{1}{\mathcal{J}} = L \underset{1 \times 6}{\mathcal{Q}} \underset{2}{\mathcal{J}}$$



$$\{F\}_{6 \times 1} = \left\{ \begin{array}{l} R_{x1} + \frac{pL}{2} \cdot s \\ R_{y1} - \frac{pL}{2} \cdot c \\ M_{z1} - \frac{pL^2}{12} \\ \frac{pL}{2} \\ R_{y2} - \frac{pL}{2} \cdot c \\ \frac{pL^2}{12} \end{array} \right\}$$

$$[k]_1 = \begin{bmatrix} a & 0 & 0 & -a & 0 & 0 \\ 0 & b & d & 0 & -b & d \\ 0 & d & m & 0 & -d & r \\ -a & 0 & 0 & a & 0 & 0 \\ 0 & -b & -d & 0 & b & -d \\ 0 & d & r & 0 & -d & m \end{bmatrix}$$

$$a = \frac{EA}{l}$$

$$b = \frac{12EJ_2}{l^3}$$

$$d = \frac{6EJ_2}{l^2}$$

$$m = \frac{4EJ_2}{l}$$

$$r = \frac{2EJ_2}{l}$$

$$[T]_1 = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

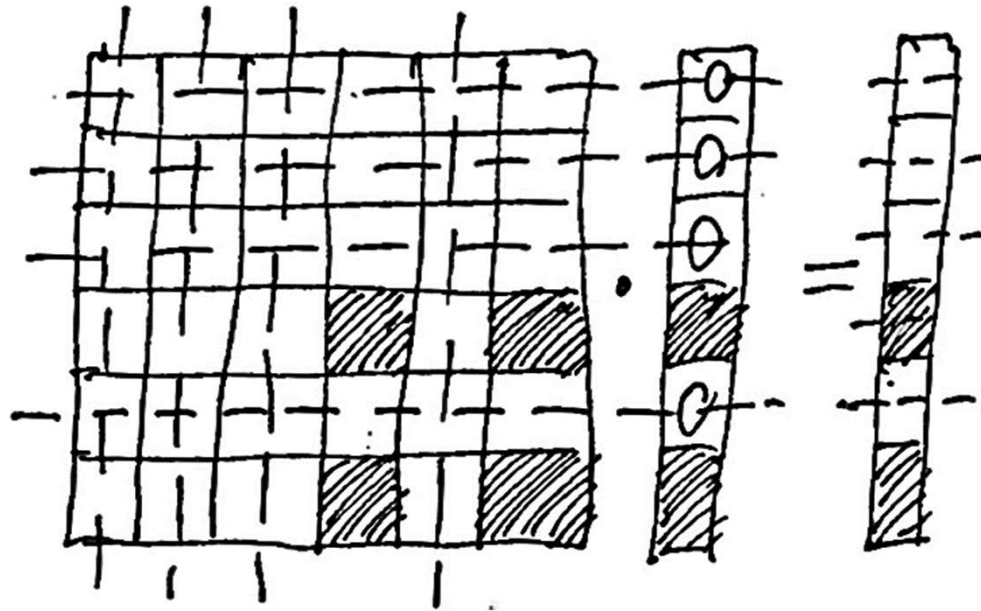
$$[T]_1^T = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[K] = [K_g]_1 = [T_f]_1^T [K]_1 [T_f]_1$$

6×6 6×6 6×6 6×6 6×6

$$[K] \cdot \{q\} = \{F\} + \text{boundary conditions:}$$

6×6 6×1 6×1 $u_1 = 0, v_1 = 0, \theta_1 = 0, v_2 = 0$



$$\begin{bmatrix} K_{44} & K_{46} \\ K_{64} & K_{66} \end{bmatrix} \cdot \begin{Bmatrix} u_2 \\ \varphi_2 \end{Bmatrix} = \begin{Bmatrix} \frac{P_0 L}{2} \\ \frac{P_0 L^2}{12} \end{Bmatrix}$$

$$\rightarrow u_2 = \frac{\frac{3}{8 S_1} P_0 L}{\frac{EA}{L \cdot t^2} + \frac{3EJ_2}{L^3}}$$

$$\varphi_2 = \frac{P_0 L^3}{48EJ_2} \left(1 - \frac{27}{\frac{AL^2}{J_2 \cdot t^2} + 3} \right)$$

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_1 = \begin{bmatrix} T_f \\ \end{bmatrix}_1 \cdot \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_1$$

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} = \begin{bmatrix} T_f \\ \end{bmatrix}_1 \cdot \begin{Bmatrix} 0 \\ 0 \\ 0 \\ u_2 \\ 0 \\ \theta_2 \end{Bmatrix}$$

$$q_1 = 0, \quad q_2 = 0, \quad q_3 = 0$$

$$q_4 = \frac{\frac{3}{8} p_0 l}{\frac{EA}{Lt^2} + \frac{3EJ_2}{L^3}}$$

$$q_5 = \frac{-\frac{3}{8} p_0 l}{\frac{EA}{Lt^2} + \frac{3EJ_2}{L^3}}$$

$$q_6 = \frac{p_0 l^3}{48EJ_2} \left(1 - \frac{27}{\frac{Al^2}{J_2 t^2} + 3} \right)$$

(the same as those obtained in 1°)

REACTIONS:

$$[K]_{6 \times 6} \cdot \{q\}_{6 \times 1} = \{F\}_{6 \times 1}$$

$$k_{14} \cdot u_2 + k_{16} \cdot \theta_2 = R_{x1} + \frac{P \cdot l}{2} \cdot s$$

$$k_{24} \cdot u_2 + k_{26} \cdot \theta_2 = R_{y1} - \frac{P \cdot l}{2} \cdot c$$

$$k_{34} \cdot u_2 + k_{36} \cdot \theta_2 = M_{z1} - \frac{P \cdot l^2}{12}$$

$$k_{54} \cdot u_2 + k_{56} \cdot \theta_2 = R_{y2} - \frac{P \cdot l}{2} \cdot c$$

$$R_{x1} = k_{14} u_2 + k_{16} \theta_2 - \frac{P \cdot l}{2} \cdot s$$

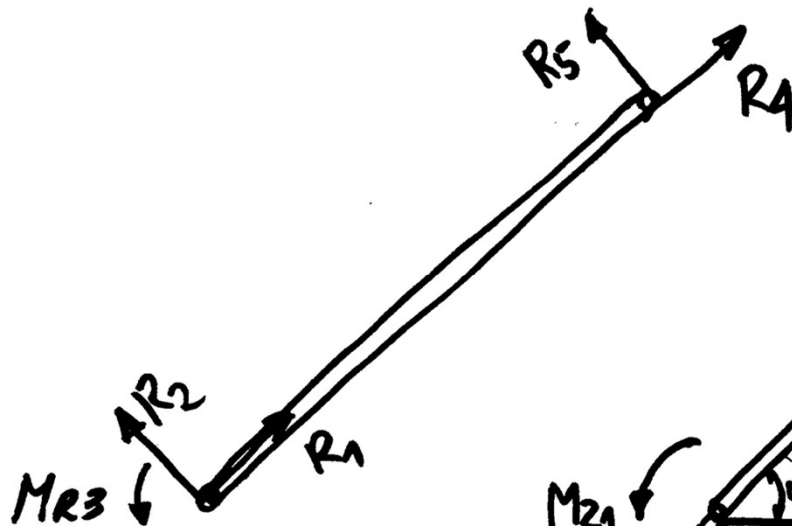
$$R_{y1} = k_{24} u_2 + k_{26} \theta_2 + \frac{P \cdot l}{2} \cdot c$$

$$M_{z1} = k_{34} u_2 + k_{36} \theta_2 + \frac{P \cdot l^2}{12}$$

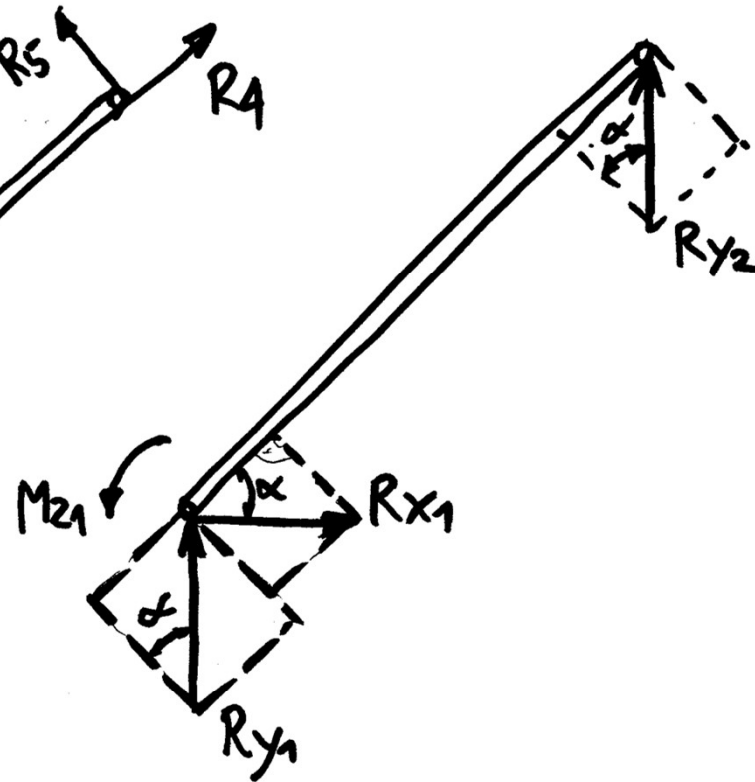
$$R_{y2} = k_{54} u_2 + k_{56} \theta_2 + \frac{P \cdot l}{2} \cdot c$$

REACTIONS

1°)



2°)



$$R_{x1} \cdot \cos \alpha + R_{y1} \cdot \sin \alpha = R_1$$

$$-R_{x1} \cdot \sin \alpha + R_{y1} \cdot \cos \alpha = R_2$$

$$R_{y2} \cdot \sin \alpha = R_4$$

$$R_{y2} \cdot \cos \alpha = R_5$$

$$M_{21} = M_3$$